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Laminar convective heat transfer of a Bingham plastic in a circular pipe—I. Analytical approach—thermally fully developed flow and thermally developing flow (the Graetz problem extended)

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Abstract—Thermally fully developed and thermally developing laminar flows of a Bingham plastic in a circular pipe have been studied analytically. For thermally fully developed flow, the Nusselt numbers and temperature profiles are presented in terms of the yield stress and Peclet number, proposing a correlation formula between the Nusselt number and the Peclet number. The solution to the Graetz problem has been obtained by using the method of separation of variables, where the resulting eigenvalue problem is solved approximately by using the method of weighted residuals. The effects of the yield stress, Peclet and Brinkman numbers on the Nusselt number are discussed. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The problem of laminar forced convective heat transfer in pipe flows is of foremost importance to the design of practical thermal systems. A large number of fluids used extensively in industrial applications exhibit yield stresses that must be overcome before they start to flow, which are called Bingham plastic. Some examples are electrorheological fluids, suspensions, drilling muds, paints, greases, aqueous foams, slurries and food products like margarine, mayonnaise and ketchup etc. The motivation of the present paper is to better understand the flow and heat transfer characteristics of such fluids.

The first studies of heat transfer in a duct flow of Newtonian fluid were made more than a century ago by Graetz in 1883–1885 and independently by Nusselt in 1910. Solutions to this problem and its various extensions continue to evoke many research efforts [1–8] and a comprehensive review of the works on heat transfer in laminar duct flow was compiled by Shah and London [9].

For a laminar Newtonian flow, it is well known that the Nusselt number for fully developed flow is 3.6568 excluding axial conduction. Pahor and Strand [2] studied thermally fully developed flow including axial conduction for a laminar Newtonian flow by the perturbation method and presented graphically the fully developed Nusselt number with respect to the Peclet number. Their work was further refined by Ash and Heinbockel [5], who considered only the first eigenmode of the eigenvalue problem originated from the Graetz problem to obtain the fully developed Nusselt number with respect to the Peclet number.

Kakac *et al.* [10] and Kays and Crawford [11] reported that Bhatti [12] obtained a fully developed temperature profile for a Newtonian fluid neglecting axial conduction and viscous dissipation. However, the reported solution is found to involve some errors, as will be shown later in the present study. Vradis *et al.* [13] reported a fully developed temperature profile for a Bingham plastic including viscous dissipation. When viscous dissipation is included and axial conduction is excluded under uniform wall temperature boundary condition, a fully developed temperature profile is obtained asymptotically. To the authors' knowledge, there is no existing solution for thermally developed flow of a Bingham plastic in laminar pipe flow including axial conduction in the case of negligible viscous dissipation.

Wissler and Schechter [14] solved the Graetz problem for a Bingham plastic neglecting both axial conduction and viscous dissipation by using the method of separation of variables, which led to a Sturm–Liouville eigenvalue problem. They numerically obtained the first seven eigenvalues and eigenfunctions for $c = 0.0, 0.25, 0.5, 0.75$ and 1.0. On the other hand, the Leveque solution of a Bingham plastic was given by Beek and Eggink [15]. Later, Blackwell [16] indicated that the number of eigenvalues and eigenfunctions obtained by Wissler and Schechter were inadequate for small values of x^+ and extended the calculations to include the first 60 eigenvalues for $c = 0.0, 0.2, 0.4, 0.8$ and 1.0.

Recently, Johnston [17] solved this problem by an approximate solution method based on the Sturm–Liouville transform theory and extended his solution

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NOMENCLATURE

A_1, A_2 parameters for correlation formula (see equation (15))
A, **B**, **C** matrices
A_{ji}, B_{ji}, C_{ji} elements of matrices **A**, **B** and **C**
a_n, b_n coefficients of infinite power series
Br Brinkman number, $\mu_0 U_{av}^2/k(T_e - T_w)$
C_n, D_n, E_n coefficients used in the solution of eigenvalue problem
C_p specific heat at constant pressure
c dimensionless radius of the plug flow region, or ratio of the yield shear stress to the wall shear stress, τ_y/τ_w
c_1, c_2 coefficients of the Bessel function solution
c, **d** eigenvectors
c_i, d_i components of eigenvectors
D pipe diameter
h heat transfer coefficient based on bulk temperature
J_0, J_1 Bessel functions of the first kind of orders 0 and 1
k thermal conductivity
N number of finite eigenmodes
Nu local Nusselt number, hD/k
Nu_0 asymptotic Nu for thermally fully developed flow when $Pe \to 0$
Nu_∞ asymptotic Nu for thermally fully developed flow when $Pe \to \infty$
Pe Peclet number, $2U_{av}R/\alpha$
Pr Prandtl number, $\mu_0/(\rho\alpha)$
R pipe radius
R_n eigenfunction
r dimensionless radial coordinate, y/R
S_i trial function
s_1, s_2, s_3, s_4, s_5 constants used in thermally fully developed flow solutions
$T(y, z)$ temperature
T_e entrance temperature
$T_m(z)$ bulk temperature
T_w wall temperature
U axial velocity
U_{av} average axial velocity
u dimensionless axial velocity, U/U_{av}
w_i weight function used in the method of weighted residuals
x^+ dimensionless axial coordinate, $(z/R)/Pe$
Y_0 Bessel function of the second kind of order 0
y radial coordinate
z axial coordinate.

Greek symbols
α thermal diffusivity
β_1 first zero of the Bessel function J_0
η apparent viscosity for a Bingham model
η_{eff} dimensionless apparent viscosity, η/μ_0
$\Theta(r, x^+)$ dimensionless temperature $(T_w - T)/(T_w - T_e)$
$\Theta_m(x^+)$ dimensionless bulk temperature $(T_w - T_m)/(T_w - T_e)$
$\Theta_\infty(r)$ dimensionless temperature for $x^+ \to \infty$
λ_n eigenvalue
μ_0 plastic viscosity
ρ density
τ shear stress
τ_w wall shear stress
τ_y yield shear stress
Φ viscous dissipation function
$\phi(r)$ dimensionless temperature $(T_w - T)/(T_w - T_m)$.

Subscript
e entrance
w wall.

to the case including only axial conduction. He concluded that the Peclet number had to be larger than 1000 in order for axial conduction term to be neglected without loss of accuracy. As far as we know, there has been no work in the literature that studied the Graetz problem for a Bingham plastic pipe flow including both axial conduction and viscous dissipation.

The objectives of the present study are two-fold. Firstly, we are to study thermally fully developed flow of a Bingham plastic including axial conduction, to obtain the Nusselt number and temperature profiles and to propose a correlation formula between the Nusselt number and the Peclet number (Section 2). Secondly, we are to study thermally developing flow (the Graetz problem) of a Bingham plastic including both axial conduction and viscous dissipation. The solution to this problem is obtained by using the method of separation of variables, where the resulting eigenvalue problem is solved approximately by using the method of weighted residuals (Section 3).

2. THERMALLY FULLY DEVELOPED FLOW

2.1. *Governing equations*

Assuming that the velocity field is fully developed and viscous dissipation is negligible, the non-dimensionalized energy equation for a circular pipe flow can be represented as

$$\frac{u}{2}\frac{\partial\Theta}{\partial x^+} = \frac{1}{Pe^2}\frac{\partial^2\Theta}{\partial x^{+2}} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Theta}{\partial r}\right) \quad (1)$$

where $\Theta(r, x^+) = (T - T_w)/(T_e - T_w)$ is the non-dimensionalized temperature, $r = y/R$, $x^+ = (z/R)/Pe$, $Pe = 2U_{av}R/\alpha$ is the Peclet number, R is the pipe radius, α is the thermal diffusivity, T_e is the entrance temperature and T_w is the wall temperature.

For constant properties, the fully developed velocity profile for a laminar pipe flow of a Bingham plastic is given as follows [18]:

$$u(r) = \begin{cases} \dfrac{2(1-c)^2}{1-\frac{4}{3}c+\frac{c^4}{3}} & 0 \leqslant r \leqslant c \\ \dfrac{2(1-r^2-2c(1-r))}{1-\frac{4}{3}c+\frac{c^4}{3}} & c \leqslant r \leqslant 1 \end{cases} \quad (2)$$

where $r = 0$ and $r = 1$ correspond, respectively, to the centerline and the wall, and c is the dimensionless radius of the plug-flow region. Thus, $c = 1$ corresponds to a complete plug flow ($u = 1$), while $c = 0$ corresponds to a laminar Newtonian flow.

The thermally fully developed condition in laminar pipe flow is defined [11] by

$$\frac{\partial}{\partial z}\left(\frac{T_w - T}{T_w - T_m}\right) = 0, \quad (3)$$

which means that the dimensionless temperature based on the difference between the wall and bulk temperatures is invariant with the axial distance of the pipe. By defining a new dimensionless temperature

$$\phi(r) = (T_w - T)/(T_w - T_m) = \Theta(r, x^+)/\Theta_m(x^+),$$

equation (1) can be rewritten as

$$\frac{u}{2}\Theta_m'\phi = \frac{1}{Pe^2}\Theta_m''\phi + \frac{1}{r}\Theta_m\phi' + \Theta_m\phi'' \quad (4)$$

where primes denote differentiation in terms of argument.

In order that a solution may be obtained by the method of separation of variables, we set $\Theta_m'/\Theta_m = -s_1 (s_1 > 0)$, which assures that $\Theta_m''/\Theta_m = s_1^2$. Then equation (4) becomes

$$\phi'' + \frac{1}{r}\phi' + \left(\frac{s_1^2}{Pe^2} + s_1\frac{u}{2}\right)\cdot\phi = 0 \quad (5a)$$

with the boundary conditions

$$\phi(1) = 0 \quad (5b)$$

$$\phi'(0) = 0 \quad (5c)$$

and from the definition of ϕ, the Nusselt number is represented as follows:

$$-2\cdot\phi'(1) = Nu. \quad (5d)$$

Integrating equation (5a) over the interval $0 \leqslant r \leqslant 1$, we have

$$\int_0^1 (r\phi')'\,dr = -\int_0^1 \left(\frac{s_1^2}{Pe^2} + s_1\frac{u}{2}\right)\phi r\,dr \quad (6)$$

$$Nu = \frac{2s_1^2}{Pe^2}\int_0^1 \phi r\,dr + \frac{s_1}{2}. \quad (7)$$

Equation (7) can be obtained by non-dimensionalizing the energy conservation relation which is expressed for a cross-section of a pipe as

$$\rho C_p U_{av}\pi R^2\frac{dT_m}{dz} = 2\pi\int_0^R k\frac{\partial^2 T}{\partial z^2}y\,dy + h(T_w - T_m)\cdot 2\pi R. \quad (8)$$

2.2. *Results and discussion*

Given the solution of the velocity field (equation (2)), the solution to the problem (5) can be obtained in the form of a Bessel function for $c = 1$, while it can be obtained in the form of an infinite power series by using the Frobenius' method for $c = 0$. For $0 < c < 1$, the solution may be obtained in a combined form of a Bessel function for $0 \leqslant r \leqslant c$ and an infinite power series for $c \leqslant r \leqslant 1$. Hence the solution of equation (5) may be expressed as

$$\phi(r) = \begin{cases} c_1 J_0(\sqrt{s_2}\cdot r) + c_2 Y_0(\sqrt{s_2}\cdot r) & 0 \leqslant r \leqslant c \\ \sum_{n=0}^{\infty} a_n r^n + \ln(r)\cdot\sum_{n=0}^{\infty} b_n r^n & c \leqslant r \leqslant 1 \end{cases} \quad (9a)$$

$$s_2 = \frac{s_1^2}{Pe^2} + \frac{s_1\cdot(1-c)^2}{1-\frac{4}{3}c+\frac{c^4}{3}} \quad (9b)$$

where the recurrence formula for a_n and b_n are

$$b_1 = a_1 = 0 \quad (10a)$$

$$b_2 = -\frac{1}{2^2}\cdot s_3\cdot b_0 \quad a_2 = -\frac{1}{2^2}(2\cdot 2b_2 + s_3\cdot a_0) \quad (10b)$$

$$b_3 = -\frac{1}{3^2}\cdot s_4\cdot b_0 \quad a_3 = -\frac{1}{3^2}(2\cdot 3b_3 + s_4\cdot a_0) \quad (10c)$$

$$b_n = -\frac{1}{n^2}(s_5\cdot b_{n-4} + s_4\cdot b_{n-3} + s_3\cdot b_{n-2})$$

$$a_n = -\frac{1}{n^2}(2n\cdot b_n + s_5\cdot a_{n-4} + s_4\cdot a_{n-3} + s_3\cdot a_{n-2}) \quad n \geqslant 4 \quad (10d)$$

and

$$s_3 = \frac{s_1^2}{Pe^2} + \frac{s_1\cdot(1-2c)}{1-\frac{4}{3}c+\frac{c^4}{3}} \quad (11a)$$

Table 1. Parameters a_0, b_0, c_1 and Nu_∞ for various values of c

c	a_0	b_0	c_1	Nu_∞
0.0	1.80261846	0.0	0.0	3.65679346
0.2	1.80442580	−2.44450615e−3	1.81100734	3.81250009
0.4	1.71572615	−6.19434447e−2	1.84010639	4.08075955
0.6	7.12456411e−1	−7.09384401e−1	1.91593691	4.49320147
0.8	−5.38076252e+1	−2.68651238e+1	2.06540621	5.06575119
1.0	0.0	0.0	2.31612940	5.78318596

$$s_4 = \frac{2 \cdot s_1 \cdot c}{1 - \frac{4}{3}c + \frac{c^4}{3}} \tag{11b}$$

$$s_5 = -\frac{s_1}{1 - \frac{4}{3}c + \frac{c^4}{3}}. \tag{11c}$$

Clearly, $c_2 = 0$ because the temperature is bounded at $r = 0$. Then equation (9) automatically satisfies the boundary condition at $r = 0$ (equation (5c)) because the derivative of J_0 is zero there. Now, we have five undetermined parameters (c_1, s_1, a_0, b_0 and Nu) and five conditions, which consist of two boundary conditions (equations (5b) and (5d)), the energy conservation (equation (7)) and two matching conditions for $\phi(r)$ and $\phi'(r)$ at $r = c$.

When $Pe \to \infty$, axial conduction can be neglected and $s_1 = 2Nu$ from equation (7). Four parameters c_1, a_0, b_0 and Nu_∞ for various values of c are presented in Table 1 and the variation of fully developed temperature profile with respect to c is shown in Fig. 1 for this case. According to Kakac *et al.* [10] and Kays and Crawford [11], Bhatti [12] obtained fully developed temperature profile for a Newtonian fluid ($c = 0$) neglecting axial conduction ($Pe \to \infty$). They presented the solution in the form of an infinite power series, which includes Bhatti's wrong data, $a_0 = 1$. The right value should be $a_0 = 1.8026$ as shown in Table 1 of the present paper. This can be readily verified from Fig. 1, by comparing the Graetz solutions of Newtonian flow [3, 4] for large x^+.

When $Pe \to 0$, one can obtain an asymptotic solution by the perturbation method similar to that used for Newtonian fluid by Pahor and Strand [2] and Michelsen and Villadsen [7]. The result can be written as

$$\phi(r) = \frac{J_0(\beta_1 r)}{2\int_0^1 uJ_0(\beta_1 r) r\,dr} \tag{12}$$

$$Nu(Pe) = Nu_0 - C_1 Pe \tag{13}$$

where $\beta_1 = 2.4048255577$ is the first zero of the Bessel function J_0, Nu_0 is a limiting value at $Pe = 0$ and C_1 is a constant. Figure 2 shows fully developed tem-

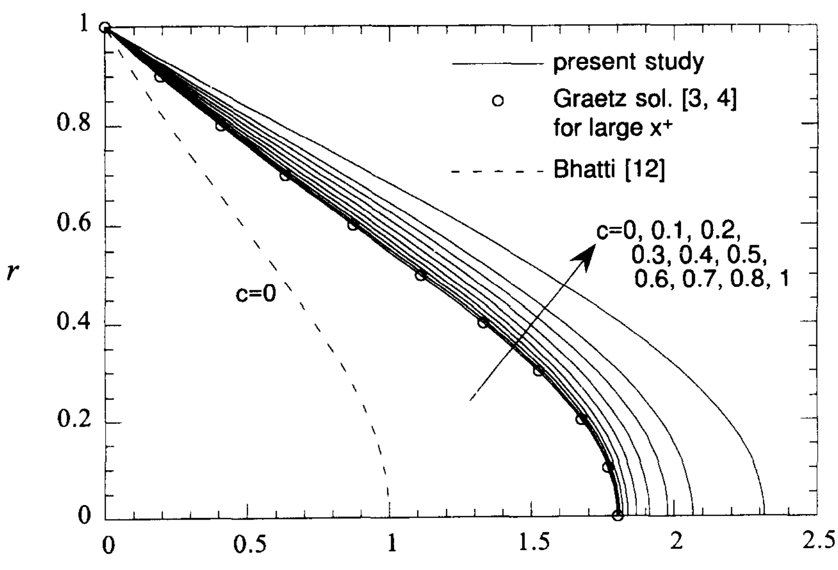


Fig. 1. Variation of fully developed temperature profile with respect to c when $Pe \to \infty$.

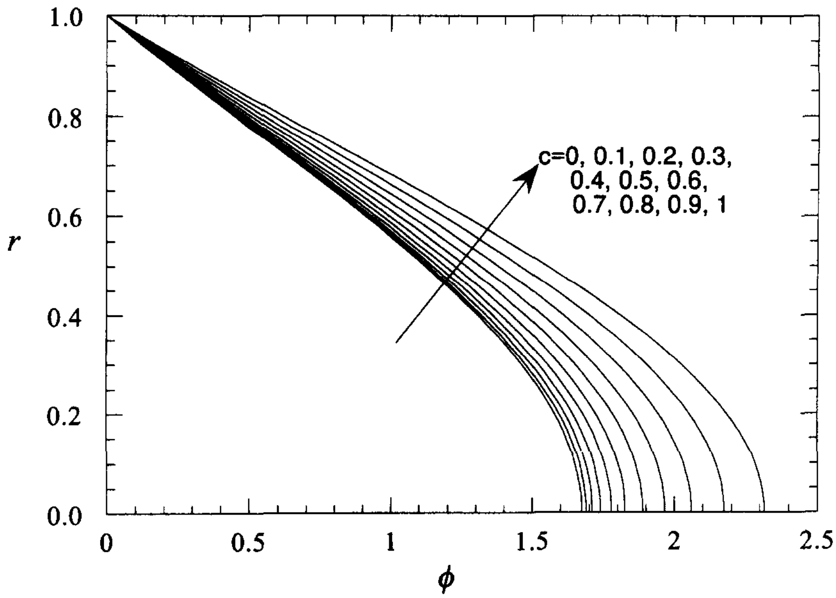


Fig. 2. Variation of fully developed temperature profile with respect to c when $Pe \to 0$.

perature profiles represented by equation (12). From equations (5d) and (12), Nu_0 can be obtained as

$$Nu_0 = \frac{\beta_1 J_1(\beta_1)}{\int_0^1 uJ_0(\beta_1 r) r \, dr}. \quad (14)$$

Using the method of Churchill and Usagi [19], an explicit correlation formula between Nu and Pe can be written as

$$Nu = Nu_0 - (Nu_0 - Nu_\infty) \cdot (1 + (A_1 \cdot Pe)^{A_2})^{1/A_2} \quad (15)$$

where A_1 and A_2 are obtained by the regression analysis. Parameters Nu_0, A_1 and A_2 for various values of c are given in Table 2.

Figure 3 shows the variation of Nu with respect to Pe for $c = 0$, 0.2, 0.4, 0.6, 0.8 and 1.0. When $c = 0$ (Newtonian fluid), the present exact solution shows good agreement with that of Ash and Heinbockel [5]. A relative error between the exact solution and prediction of Nu using a correlation formula (15) does not exceed 0.1%. It is also shown that the present correlation formula predicts the exact solution much better than that of Michelsen and Villadsen [7] for the entire range of Pe, who derived the correlation formula by using the perturbation methods both when Pe is very large and when Pe is very small. For larger c, the variation of Nu with respect to Pe decreases, until for $c = 1$ Nu is completely independent of Pe.

Table 2. Parameters Nu_0, A_1 and A_2 for various values of c

c	Nu_0	A_1	A_2
0.0	4.18065498	0.31362065	−1.66141012
0.2	4.27501534	0.29957298	−1.66123843
0.4	4.44229997	0.27736639	−1.66306874
0.6	4.71804885	0.24787100	−1.66891216
0.8	5.14500676	0.21563880	−1.67724511

Figures 4(a)–(e) show the temperature profiles for $c = 0, 0.2, 0.4, 0.6$ and 0.8, respectively. The variation of the temperature profiles with respect to Pe is again shown to be smaller for larger c. The temperature profile for $c = 1$ is not shown because it is the same as Figs 1 and 2 and does not vary with Pe.

3. THERMALLY DEVELOPING FLOW—THE GRAETZ PROBLEM

3.1. *Governing equations*

The non-dimensionalized governing equation for the Graetz problem in laminar pipe flow including both axial conduction and viscous dissipation can be represented as

$$\frac{u}{2}\frac{\partial \Theta}{\partial x^+} = \frac{1}{Pe^2}\frac{\partial^2 \Theta}{\partial x^{+2}} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right) + Br \cdot \eta_{\text{eff}} \cdot \Phi \quad (16)$$

$$\eta_{\text{eff}} = \begin{cases} \infty & 0 \leqslant r \leqslant c \\ 1 + \dfrac{\tau_y R/\mu_0 U_{\text{av}}}{|\mathrm{d}u/\mathrm{d}r|} & c \leqslant r \leqslant 1 \end{cases} \quad (17)$$

where the viscous dissipation, represented by $\eta_{\text{eff}} \cdot \Phi$, is a known function because the velocity profile is again assumed to be fully developed (equation (2)) and $\Phi = (\mathrm{d}u/\mathrm{d}r)^2$.

For uniform wall and inlet temperatures, the boundary conditions are written as

$$\Theta(r, 0) = 1, \quad (18a)$$

$$\Theta(1, x^+) = 0, \quad \left.\frac{\partial \Theta}{\partial r}\right|_{r=0} = 0 \quad (18b)$$

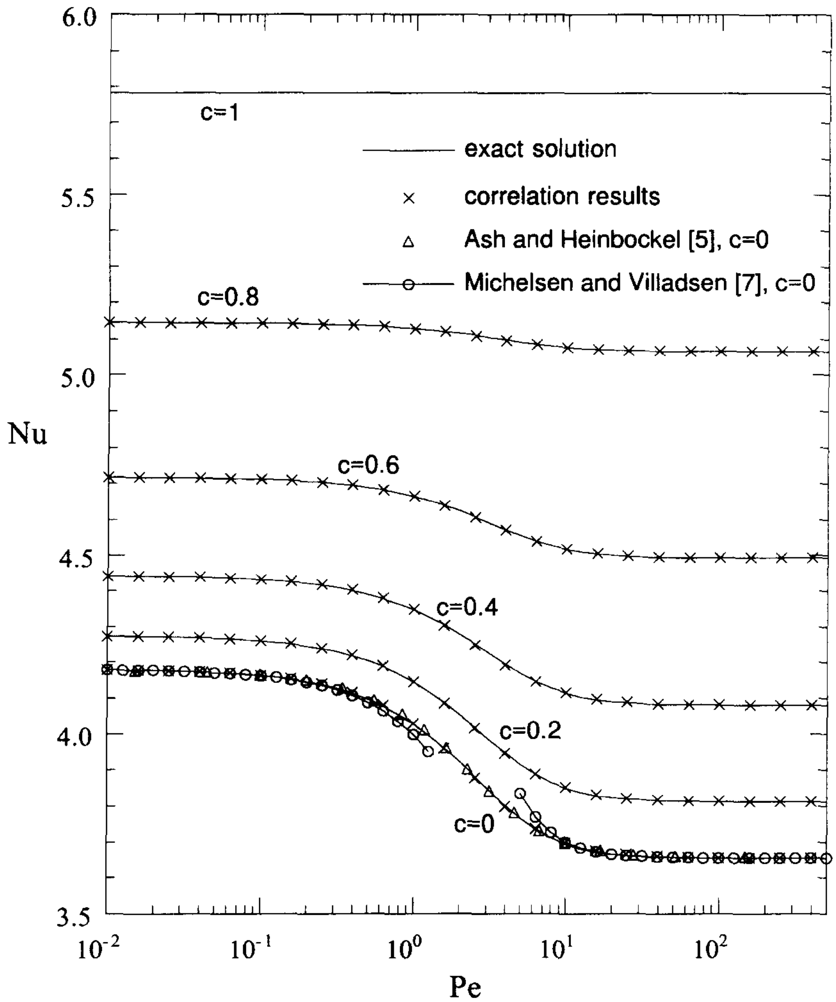


Fig. 3. Variation of the Nusselt number with respect to the Peclet number for $c = 0, 0.2, 0.4, 0.6, 0.8$ and 1.

$$\lim_{x^+ \to \infty} \Theta(r, x^+) = \Theta_\infty(r) \tag{18c}$$

where $\Theta_\infty(r)$ is a solution of

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Theta_\infty}{\mathrm{d}r}\right) + Br \cdot \eta_{\mathrm{eff}} \cdot \Phi = 0 \tag{19a}$$

$$\Theta_\infty(1) = 0, \quad \Theta'_\infty(0) = 0. \tag{19b}$$

The solution of equation (19) was obtained by Vradis *et al.* [13] such that

$$\Theta_\infty(r) = \begin{cases} \dfrac{Br \cdot \left((1-c^4) - \dfrac{16c}{9}(1-c^3) - \dfrac{4c^4}{3}\ln c\right)}{\left(1 - \dfrac{4c}{3} + \dfrac{c^4}{3}\right)^2} & 0 \leqslant r \leqslant c \\ \dfrac{Br \cdot \left((1-r^4) - \dfrac{16c}{9}(1-r^3) - \dfrac{4c^4}{3}\ln r\right)}{\left(1 - \dfrac{4c}{3} + \dfrac{c^4}{3}\right)^2} & c \leqslant r \leqslant 1 \end{cases} \tag{20}$$

3.2. *Method of separation of variables*

We set the solution of equation (16) in the following form

$$\Theta(r, x^+) = \Theta_\infty(r) + X(x^+) \cdot R(r). \tag{21}$$

Substituting equation (21) into equation (16) gives

$$\frac{u}{2}X'R = \frac{1}{Pe^2}X''R + \frac{1}{r}XR' + XR'' + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Theta_\infty}{\mathrm{d}r}\right) + Br \cdot \eta_{\mathrm{eff}} \cdot \Phi. \tag{22}$$

From equation (19a), equation (22) becomes

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{u}{2}\frac{X'}{X} - \frac{1}{Pe^2}\frac{X''}{X}. \tag{23}$$

In order that a solution may be obtained by the method of separation of variables, we set $X'/X = -\lambda (\lambda > 0)$, which assures that $X''/X = \lambda^2$. Then, equation (23) is now written as

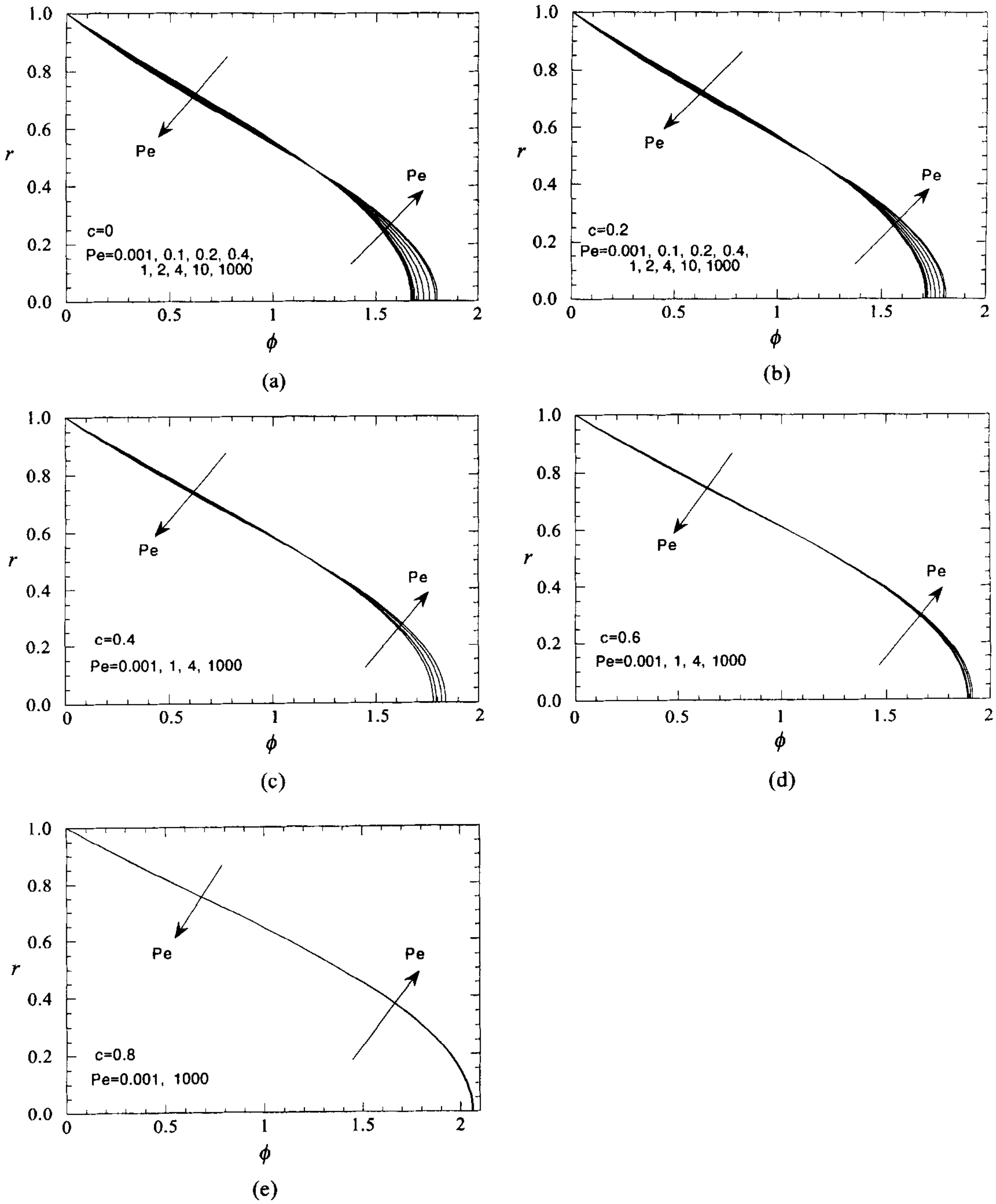


Fig. 4. Fully developed temperature profiles: (a) $c = 0$; (b) $c = 0.2$; (c) $c = 0.4$; (d) $c = 0.6$; (e) $c = 0.8$.

$$(rR')' + \left(\frac{\lambda^2}{Pe^2} + \lambda \frac{u}{2}\right) \cdot rR = 0 \tag{24a}$$

$$R'(0) = 0 \quad R(1) = 0. \tag{24b}$$

This constitutes an eigenvalue problem which is similar to the one studied earlier by Michelsen and Villadsen [7].

Thus, the solution of equation (16) can be represented as

$$\Theta(r, x^+) = \Theta_\infty(r) + \sum_{n=1}^{\infty} C_n R_n(r) \exp(-\lambda_n x^+) \tag{25}$$

where λ_n is an eigenvalue and R_n is the corresponding eigenfunction. Since the axial conduction term, $(1/Pe^2)\partial^2\Theta/\partial x^{+2}$, is retained in equation (16), the mathematical problem (24a) and (24b) does not reduce to a classical Sturm–Liouville system with a complete set of eigenfunctions orthonormal with respect to a weighting function [20].

Hence, we determine the coefficients C_n approximately by considering finite N eigenmodes (λ_n, $R_n(r)$) adequate for a converged solution. The method of weighted residuals through which these finite N eigenmodes are determined is explained in the next subsection. Applying the boundary condition at the entrance, multiplying equation (25) by R_m and integrating over $0 \leqslant r \leqslant 1$, we obtain

$$\sum_{n=1}^{N} C_n \int_0^1 R_m R_n \, dr = \int_0^1 R_m (1-\Theta_\infty) \, dr$$

$$m = 1, 2, \ldots, N. \quad (26)$$

Solving the above set of N simultaneous equations (26), we can determine C_n.

The dimensionless bulk temperature Θ_m and the local Nusselt number Nu are then determined as

$$\begin{aligned}\Theta_m &= 2\int_0^1 u\Theta r \, dr \\ &= 2\int_0^1 u\Theta_\infty r \, dr + 2\sum_{n=1}^{N} C_n \\ &\quad \times \left[\int_0^1 u R_n r \, dr\right] \exp(-\lambda_n x^+) \\ &= 2\int_0^1 u\Theta_\infty r \, dr + 2\sum_{n=1}^{N} \frac{D_n}{\lambda_n} \exp(-\lambda_n x^+) \\ &\quad + 2\sum_{n=1}^{N} \frac{E_n \lambda_n}{Pe^2} \exp(-\lambda_n x^+) \end{aligned} \quad (27)$$

$$\begin{aligned} Nu &= -\frac{2}{\Theta_m} \left.\frac{\partial \Theta}{\partial r}\right|_{r=1} \\ &= -\frac{2\Theta'_{\infty,r=1}}{\Theta_m} + \frac{1}{\Theta_m}\sum_{n=1}^{N} D_n \exp(-\lambda_n x^+) \end{aligned} \quad (28)$$

where

$$D_n = -2C_n R'_n(1), \quad E_n = -2C_n \left[\int_0^1 r R_n \, dr\right].$$

3.3. *The method of weighted residuals.*

It is well known that the construction of an exact analytical solution to the eigenvalue problem of thermally developing flow is very difficult even in the case of negligible axial conduction. Although numerical solutions to the eigenvalue problem (24) are possible, higher modes of the system for large eigenvalues necessitate extensive computational effort, as Sellars *et al.* [1] reported. Therefore, in this paper, an approximate solution to the eigenvalue problem (24) is obtained by the method of weighted residuals.

We expand the eigenfunction R_n in terms of some known functions $S_i(r)$ which satisfy the boundary conditions (equation (24b))

$$R_n = \sum_{i=1}^{N} c_i^{(n)} S_i(r), \quad n = 1, 2, \ldots, N \quad (29)$$

where $c_i^{(n)}$ are undetermined coefficients. In the present study, we set $S_i = \cos((2i-1)/2)\pi r$. Integrating equation (24a) multiplied by the weight function w_j, it can be written as

$$\int_0^1 (rR'_n)' w_j \, dr + \int_0^1 \left(\frac{\lambda_n^2}{Pe^2} + \lambda_n \frac{u}{2}\right) r R_n w_j \, dr = 0. \quad (30)$$

In the Galerkin method, $w_j = S_j$ and equation (30) becomes (for more details, see Finlayson [21]).

$$\sum_{i=1}^{N} (A_{ji} + \lambda_n^2 B_{ji} + \lambda_n C_{ji}) c_i^{(n)} = 0, \quad n = 1, 2, \ldots, N \quad (31)$$

where A_{ji}, B_{ji} and C_{ji} are elements of matrices such that

$$A_{ji} = -\int_0^1 r S'_i w'_j \, dr \quad (32a)$$

$$B_{ji} = \frac{1}{Pe^2}\int_0^1 r S_i w_j \, dr \quad (32b)$$

$$C_{ji} = \int_0^1 \frac{u}{2} r S_i w_j \, dr. \quad (32c)$$

This can be turned into a $2N \times 2N$ linear matrix problem by adding an unknown eigenvector $\mathbf{d}$

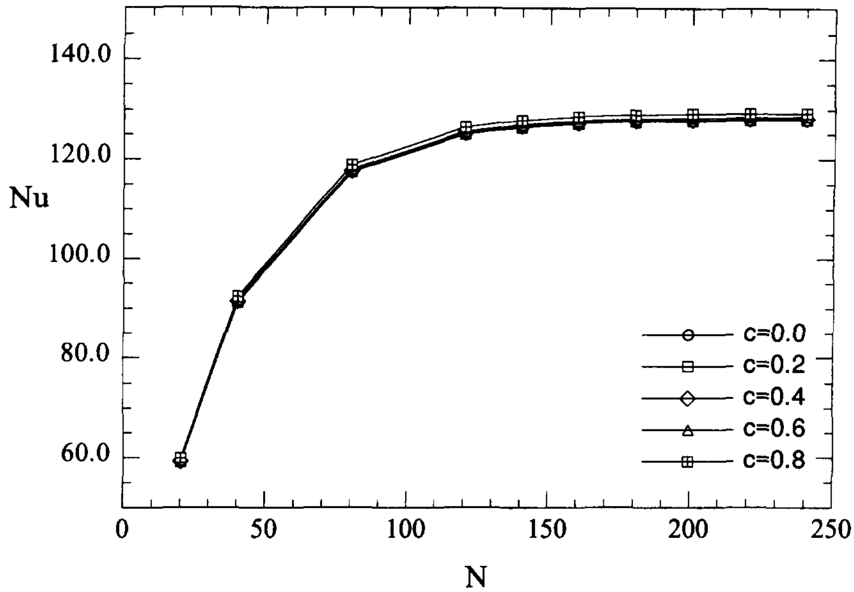
$$\begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1}\mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} \quad (33)$$

where $\mathbf{A} = \{A_{ji}\}$, $\mathbf{B} = \{B_{ji}\}$ and $\mathbf{C} = \{C_{ji}\}$. Solving equation (33), we can obtain N real positive λs and the corresponding eigenvectors.

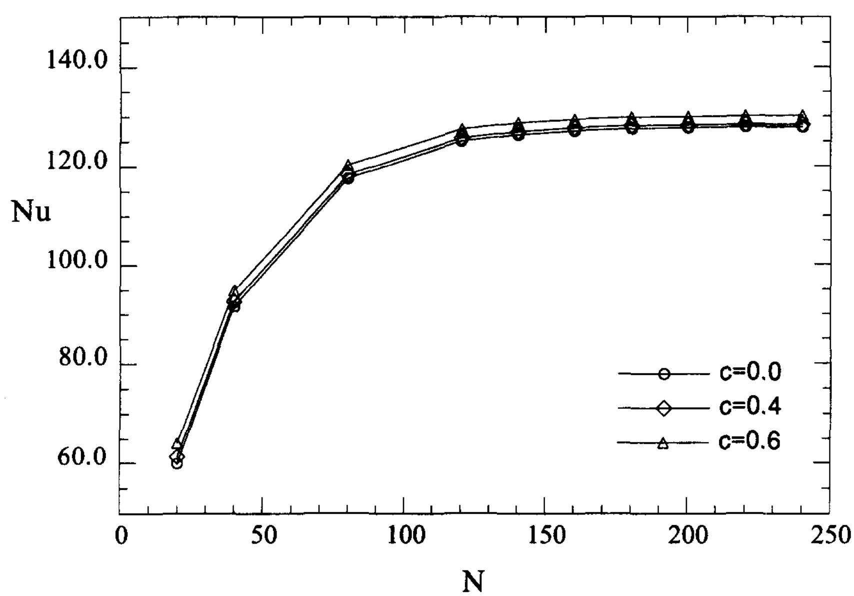
3.4. *Results and discussion*

The method of the previous subsection is applied for $Pe = 10$, 100, 1000, with $Br = 0$, 0.1, 1, 2 and $c = 0$, 0.2, 0.4, 0.6, 0.8 over the range of $0.001 \leqslant x^+ \leqslant 10$. Thermally fully developed temperature profiles in Section 2 can be readily verified by comparing the thermally developing flow solutions obtained in this section for $Br = 0$ and large x^+. The compared results are not presented here because they match very well and the differences are not discernible on figures.

Figures 5(a) and (b) show the variation of the local Nusselt number at $x^+ = 0.001$ for $Pe = 10$ with respect to the number of eigenmodes N when $Br = 0$ and $Br = 1$, respectively. Johnston [17] reported that a larger N was needed with the decrease of Pe and x^+, and with the increase of c. In the present method, however, the number N required to obtain a converged solution seems to be independent of c. At



(a) Br=0



(b) Br=1

Fig. 5. Variation of local Nusselt number at $x^+ = 0.001$ for $Pe = 10$ with respect to N: (a) $Br = 0$; (b) $Br = 1$.

$x^+ = 0.001$, about 200 eigenmodes are required to obtain a converged solution, while only 2–30 eigenmodes are required for $x^+ > 0.1$, with the convergence criterion of $(Nu^{N+1} - Nu^N)/Nu^N < 10^{-5}$.

Figures 6(a) and (b) show the local Nusselt number for $c = 0$ and the bulk temperature for $c = 0.4$, respectively, neglecting viscous dissipation ($Br = 0$). It can be seen from these figures that the present results are in good agreement with previous results [16, 17]. As expected, when Pe increases, the curves approach that of infinite Pe reported by Blackwell [16] and Johnston [17], who analyzed the problem by neglecting axial conduction. Therefore, it can be concluded that the effect of axial conduction on the bulk temperature can

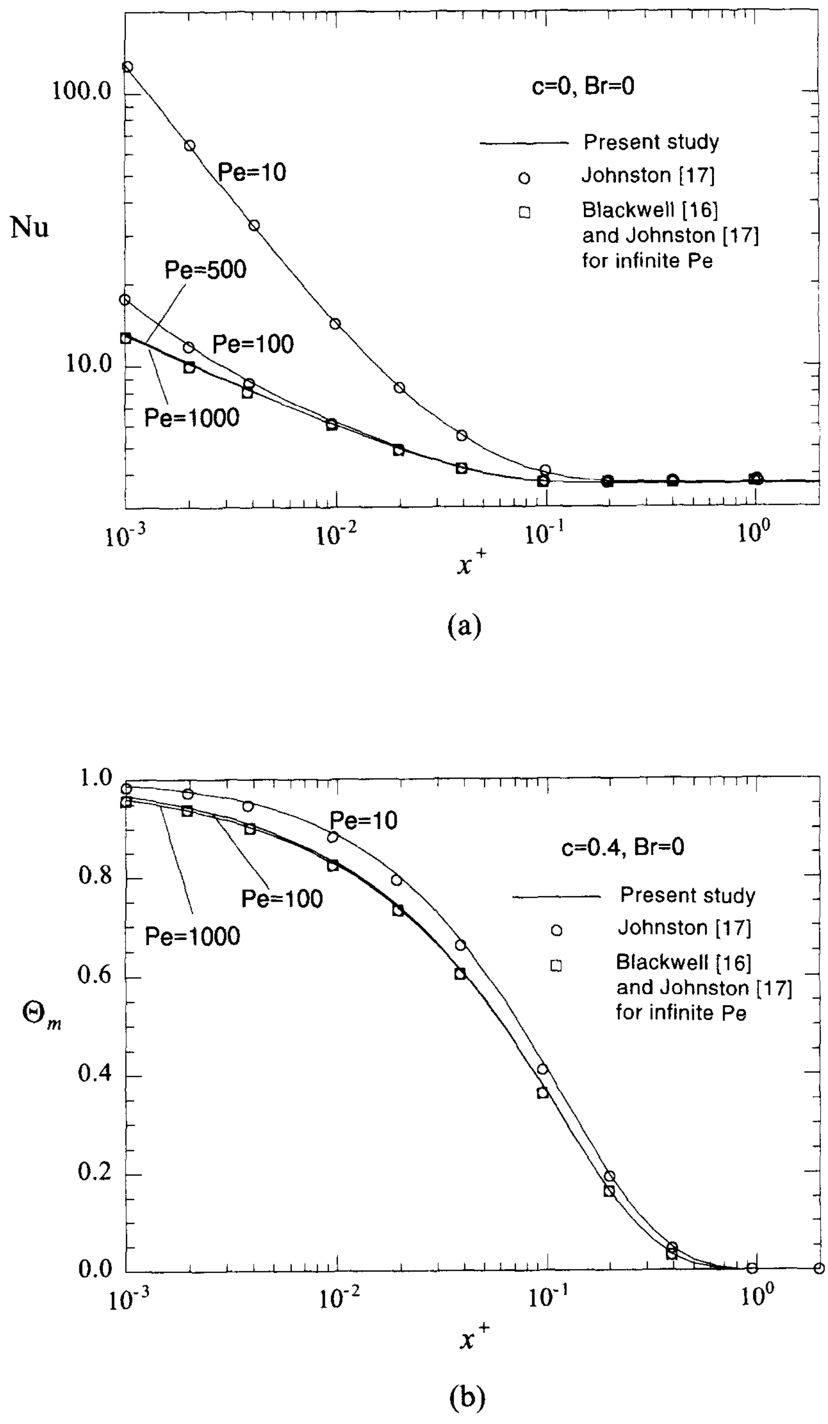


Fig. 6. (a) Local Nusselt number with respect to axial distance for $c = 0$ (excluding viscous dissipation); (b) bulk temperature with respect to axial distance for $c = 0.4$ (excluding viscous dissipation).

be neglected when Pe is larger than 100 (Fig. 6(b)), while the same argument can be made for the Nusselt number when Pe is larger than 500 (Fig. 6(a)).

Figures 7(a) and (b) show the bulk temperature and the local Nusselt number for $c = 0$, respectively, including viscous dissipation. These results also show good agreement with previous results [8]. It can be clearly seen that the Nusselt number does not decrease monotonically and there is a minimum in the Nusselt number for $Br = 0.1$ (Fig. 7(b)). This results from the fact that for some values of Br the cooling effect dominates over the viscous heating effect in the nearer entrance region, while the viscous heating effect dominates over the cooling effect in the large x^+ region [9].

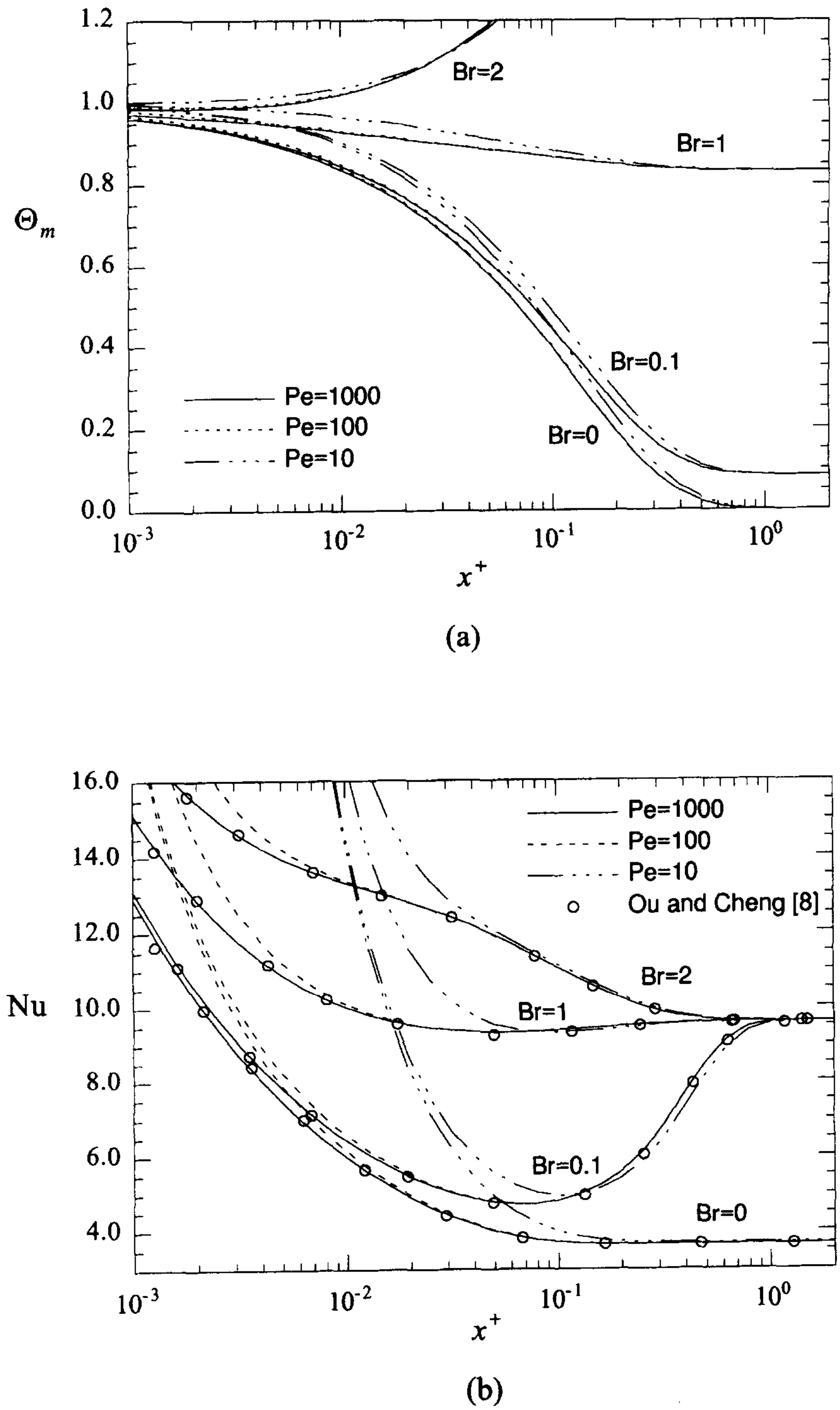


Fig. 7. (a) Bulk temperature with respect to axial distance for $c = 0$; (b) local Nusselt number with respect to axial distance for $c = 0$.

The Nusselt number variation can be explained from the variation of the temperature gradient at the wall and the local bulk temperature. Although the bulk temperature does not vary substantially with the increase of *Br* in the inlet region (Fig. 7(a)), the Nusselt number in that region considerably decreases (Fig. 7(b)). This shows that the temperature gradients significantly decrease in that region. The Nusselt number approaches the same value for all values of *Br* except for $Br = 0$ in the downstream region ($x^+ > 1$) (Fig. 7(b)).

The variation of the local Nusselt number for $Br = 0, 0.1, 1, 2$ with respect to *Pe* and *c* are shown in Fig. 8. For $Br = 0$, the local Nusselt number does

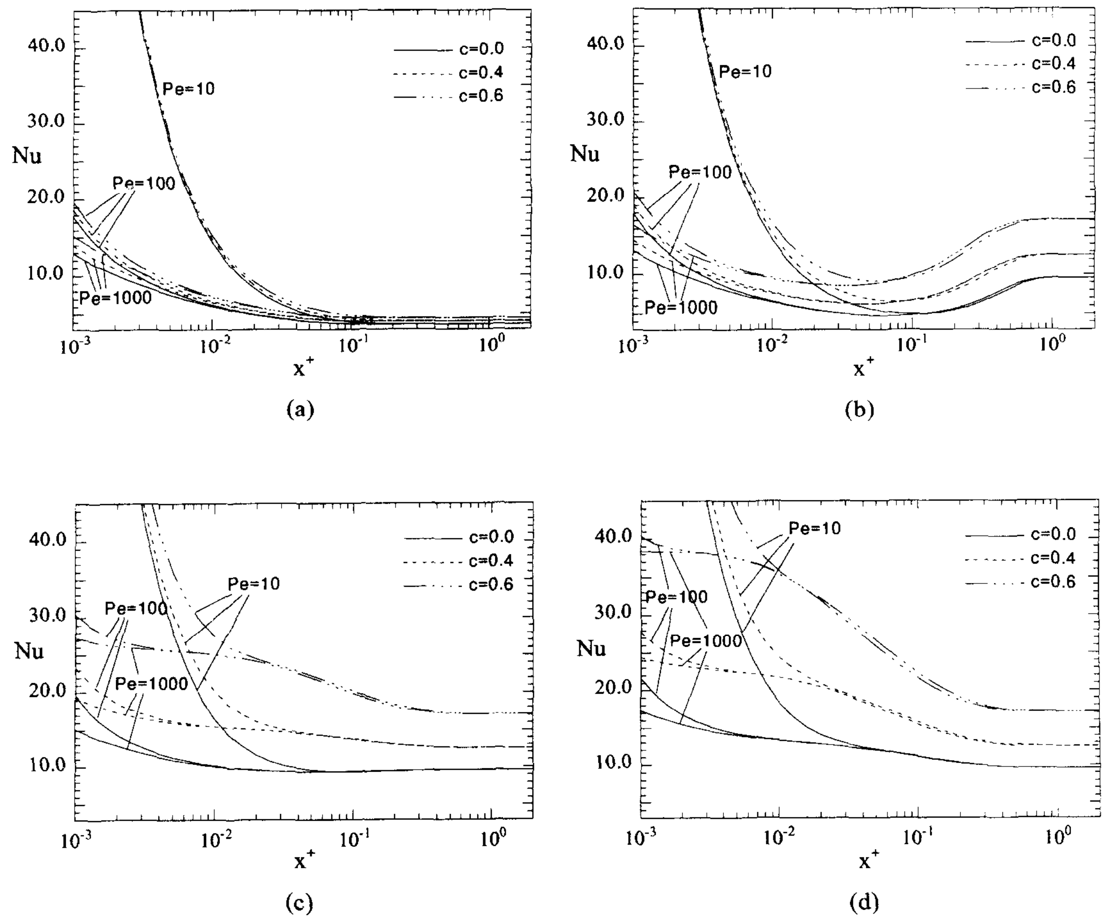


Fig. 8. Local Nusselt number with respect to axial distance: (a) $Br = 0$; (b) $Br = 0.1$; (c) $Br = 1$; (d) $Br = 2$.

not vary significantly with respect to c (Fig. 8(a)). However, when viscous dissipation is included ($Br \neq 0$), the local Nusselt number is mainly affected by c for $x^+ > 0.01$ (Figs 8(b)–(d)).

4. CONCLUSION

Thermally fully developed flow including axial conduction as well as thermally developing flow (the Graetz problem) including both axial conduction and viscous dissipation for a Bingham plastic in a laminar pipe flow have been investigated analytically in the present paper.

For thermally fully developed flow, temperature profiles were presented in a combined form of a Bessel function and an infinite power series, and the effects of the yield stress and the Peclet number on the Nusselt number were considered. It was found that the Nusselt number and the temperature profile for a Bingham plastic were affected insignificantly by the Peclet number for larger yield stress. A correlation formula between the Nusselt number and the Peclet number was proposed, which predicted the Nusselt number very well for the entire range of the Peclet number.

Analytical solution to the Graetz problem was obtained by the method of separation of variables, incorporating an approximate solution to the resultant eigenvalue problem which was obtained by using the method of weighted residuals. The Nusselt number of a Bingham plastic was not affected significantly by the yield stress when viscous dissipation was excluded. However, the Nusselt number was significantly changed by the yield stress with the inclusion of viscous dissipation. Therefore, in the case of a Bingham plastic, viscous dissipation plays a predominant role in determining the heat transfer characteristics of thermally developing flow.

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